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# Controlling high-order chaotic discrete systems by lagged adaptive adjustment

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A lagged adaptive adjustment mechanism has been developed to stabilize a high-order discrete system. Theoretical analysis and computer simulations have been provided to show the effectiveness and efficiency of this mechanism in practice.

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## I. MOTIVATIONS

In recent years there has been much attention given to theoretical and experimental methods for controlling chaos, or more generally, stabilizing unstable dynamical systems. Recent advances and developments can be seen from [1–6] and references therein. In Ref. [2], an adaptive adjustment mechanism is proposed to stabilize an unstable multidimensional discrete system. This mechanism, while inherited from the adaptive expectation scheme widely applied in economics, possesses many unique advantages over the others such as (i) demanding neither *a priori* information about the system itself nor any externally generated control signal and (ii) always forcing the original system to converge to its generic periodic points. In this paper, the same adaptive ''spirit'' is applied to the stabilization of an unstable high-order discrete system with a lagged adaptive adjustment mechanism, where adjustments are achieved through delayed feedback.

The issue of controlling high-order chaos in the framework of spatiotemporally coupled map lattice systems has long been the interest of physicists. Experimental successes have been demonstrated in various researches [3,6]. In contrast, we shall focus on high-order discrete systems in general instead of some specific systems, and emphasize on exploring the nature of unstable periodic (fixed) points. Both theoretical studies and computer simulations will be carried out to show the pros and cons of this mechanism in practice.

## II. ADAPTIVE FEEDBACK MECHANISM

Consider an $n$-dimensional dynamical system defined by

$$\mathbf{X}_{t+1}=\mathbf{F}(\mathbf{X}_t), \tag{1}$$

where $\mathbf{X}_t=(x_{1t},x_{2t},\ldots,x_{nt})$, and $\mathbf{F}=(f_1,f_2,\ldots,f_n)$, with $f_i$ being well defined functions on a domain in $\mathbb{R}^n$.

*Definition 1*: By *adaptive adjustment mechanism*, we mean the following adjusted system:

$$\mathbf{X}_{t+1}=\tilde{\mathbf{F}}_{\mathbf{\Gamma}}=(\mathbf{I}-\mathbf{\Gamma})\mathbf{F}(\mathbf{X_t})+\mathbf{\Gamma}\mathbf{X_t}, \tag{2}$$

where $\mathbf{\Gamma}=\mathrm{diag}\{\gamma_1,\gamma_2,\ldots,\gamma_n\}$ is a diagonal matrix and is referred to as an *adaptive parameter matrix*. The value of $\gamma_i$ represents the adjustment speed for $i$th variable ($i=1,2,\ldots,n$) and is assumed to vary in a *conventional range* $[0,1]$ and in a *generalized range* $[1,+\infty)$.

Let $\overline{\mathbf{X}}$ be a fixed point of Eq. (1), that is, $\overline{\mathbf{X}}=\mathbf{F}(\overline{\mathbf{X}})$. It is easy to see that the system $\tilde{\mathbf{F}}_{\Gamma}$ shares exactly the same set of fixed points of $\mathbf{F}$, that is, $\overline{\mathbf{X}}=\tilde{\mathbf{F}}_{\Gamma}(\overline{\mathbf{X}})$.

Denote $\mathcal{J}(\overline{X})$ as the Jacobian matrix of the original system $\mathbf{F}$, evaluated at $\overline{\mathbf{X}}$, with $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ as the $n$ roots of the characteristic equation, i.e.,

$$|\lambda\mathbf{I}-\mathcal{J}(\overline{\mathbf{X}})|=\prod_{j=1}^{n}(\lambda-\lambda_j)=0,$$

where $\mathbf{I}$ is an identity matrix.

The stability of a fixed point, $\overline{\mathbf{X}}$, is jointly determined by all the eigenvalues $\{\lambda_j\}$. Let $|\lambda_{\max}|=\max_j|\lambda_j|$. Mathematically, the fixed point $\overline{\mathbf{X}}$ is stable if $|\lambda_{\max}|<1$.

Denote a pair of complex conjugates $\lambda_j$ and $\overline{\lambda_j}$ by

$$\lambda_j=a_j+b_j\mathbf{i},\quad \overline{\lambda_j}=a_j-b_j\mathbf{i},$$

with the modulus $|\lambda_j|=|\overline{\lambda_j}|=\sqrt{a_j^2+b_j^2}$.

It is shown in Huang [2] that, for a $n$-dimensional dynamical system, Eq. (1), if an unstable fixed point $\overline{\mathbf{X}}$ is either a Type-I fixed point ($a_j<1$ for all $j=1,2,\ldots,n$) or a Type-II fixed point ($a_j>1$ for all $j=1,2\ldots,n$), it can always be stabilized through an adaptive adjustment mechanism with a suitable choice of adaptive parameter matrix.

Now consider a general high-order discrete system:

$$y_t=f(y_{t-1},y_{t-2},\ldots,y_{t-n}), \tag{3}$$

where $n>1$ and, without loss of generality, $f$ is assumed to be first-order continuous.

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The study of Eq. (3) is usually conducted in a multidimensional space through variable transformations

$$\begin{aligned} x_{1,t} &= y_{t-n+1} \\ x_{2,t} &= y_{t-n+2} \\ &\cdots \\ x_{n-1,t} &= y_{t-1} \\ x_{n,t} &= y_t, \end{aligned} \tag{4}$$

with which the high-order dynamical system Eq. (3) is converted into a multidimensional discrete dynamical system as follows:

$$\begin{aligned} x_{1,t+1} &= x_{2,t} \\ x_{2,t+1} &= x_{3,t} \\ &\cdots \\ x_{n-_1,t+1} &= x_{n,t} \\ x_{n,t+1} &= f(x_{n,t}, x_{n-1,t}, \ldots, x_{1,t}). \end{aligned} \tag{5}$$

Therefore, the stabilization of Eq. (3) can be achieved through the stabilization of Eq. (5). If an adaptive adjustment mechanism defined in Eq. (2) is applied, the system Eq. (5) must be modified as

$$\begin{aligned} x_{1,t+1} &= (1-\gamma_1)x_{2,t} + \gamma_1 x_{1,t} \\ x_{2,t+1} &= (1-\gamma_2)x_{3,t} + \gamma_2 x_{2,t} \\ &\cdots \\ x_{n-1,t+1} &= (1-\gamma_{n-1})x_{n,t} + \gamma_{n-1} x_{n-1,t} \\ x_{n,t+1} &= (1-\gamma_n) f(x_{n,t}, x_{n-1,t}, \ldots, x_{1,t}) + \gamma_n x_{n,t}, \end{aligned} \tag{6}$$

with adjustment parameters $\gamma_i > 0$, for $i = 1,2, \ldots, n$.

In reality, however, stabilizing the high-order system through the implementations of adaptive mechanism (6) may turn out to be either impractical or expensive. It can be both more practical and more economical to stabilize a high-order system defined by (3) with adaptive adjustment through lagged variables. Therefore, we propose the following lagged adaptive adjustment mechanism for the high-order discrete dynamical system (3).

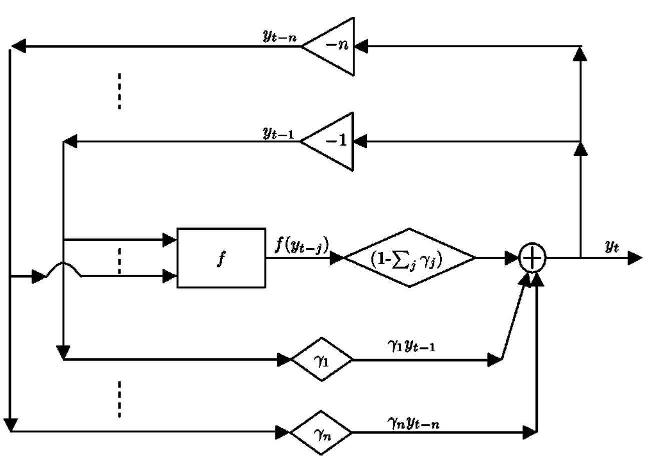


FIG. 1. Lagged adaptive adjustment.

*Definition 2*. By a *lagged adaptive adjustment* for a higher-order discrete system (3), we mean the following adjusted system:

$$\begin{aligned} y_t &= \tilde{f}(y_{t-1}, y_{t-2}, \ldots, y_{t-n}) \\ &= \left(1 - \sum_{j=1}^{n} \gamma_j\right) f(y_{t-1}, y_{t-2}, \ldots, y_{t-n}) + \sum_{j=1}^{n} \gamma_j y_{t-j}, \end{aligned} \tag{7}$$

where $\gamma_j$, $j = 1,2, \ldots, n$, are *feedback parameters*.

It is not difficult to verify that, if $\bar{y}$ is a fixed point of the high-order dynamical system (3), that is, $\bar{y} = f(\bar{y}, \bar{y}, \ldots, \bar{y})$, then $\bar{y}$ is also a fixed point of $\tilde{f}$ and vice versa.

The practical implementation of lagged adaptive adjustment [8] is illustrated in Fig. 1. While lagged variables $\{y_{t-j}\}_{j=1}^{n}$ are assumed to be available for the original system, only "additive" and "scaling" devices are utilized to achieve the adjustment. Therefore, the whole implementation requires neither a *priori* internal information about the system itself nor any externally generated control signal.

With the same transformation given by Eq. (4), the equivalent multidimensional counterpart for Eq. (7) is

$$\begin{aligned} &x_{1,t+1}=x_{2,t}\\ &x_{2,t+1}=x_{3,t}\\ &\cdots\\ &x_{n-1,t+1}=x_{n,t}\\ &x_{n,t+1}=\left(1-\sum_{j=1}^{n}\gamma_j\right)f(x_{n,t},x_{n-1,t},\ldots,x_{1,t})+\sum_{j=1}^{n}\gamma_{n-j+1}x_{j,t}. \end{aligned} \tag{8}$$

The differences in implementation between the adaptive adjustment mechanism and lagged adaptive adjustment are clearly self-evident in Eqs. (6) and (8).

### III. MAIN THEOREM

Let $f_i'$ denote the partial derivative of $f$ with respect to $y_{t-i}$ evaluated at an unstable fixed point $\overline{y}$, that is,

$$f_i'=\frac{\partial f}{\partial y_{t-i}}\bigg|_{y_{t-1}=y_{t-2}\cdots=y_{t-n}=\overline{y}}.$$

The following theorem provides a necessary and sufficient condition for the success of stabilization with lagged adaptive adjustment.

*Theorem 1.* For a high-order discrete dynamical system defined by Eq. (3), with f being first-order continuous, when and only when the condition

$$\sum_{j=1}^{n} f_j'\neq 1, \tag{9}$$

is met will there exist at least one set of feedback parameters $\{\gamma_i^*\}_{i=1}^n$ such that the unstable fixed point $\overline{y}$ can be stabilized through adaptive feedback defined by Eq. (7) when $\gamma_i\in(\gamma_i^*-\epsilon_j,\gamma_i^*+\epsilon_j)$, where $\epsilon_j\geqslant 0$, for all $i=1,2,\ldots,n$.

*Proof.* We start with the "when" part, that is, the sufficient condition.

The related Jacobian evaluated at the unstable fixed point $\overline{y}$ for Eq. (8) is given by

$$\tilde{\mathcal{J}}(\overline{y})=\begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ \vdots & \cdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 1\\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{pmatrix},$$

where

$$a_i=\left(1-\sum_{j=1}^{n}\gamma_j\right)f_i'+\gamma_i,$$

for $i=1,2,\ldots,n$.

The characteristic equation of $\tilde{\mathcal{J}}(\overline{y})$ can thus be expressed as

$$\lambda^n-a_1\lambda^{n-1}-a_2\lambda^{n-2}-\cdots-a_{n-1}\lambda-a_n=0. \tag{10}$$

The convergence of adjusted system (8) to the unstable fixed point $\overline{y}$ requires that the maximum modulus of characteristic roots for Eq. (10) must be strictly less than unity, which can be ensured if each and every $a_j$ can be forced to be arbitrarily small in absolute value by suitably choosing $\gamma_i=\gamma_i^*$, for all $i=1,2,\ldots,n$.

We thus need to show the existence of such $\{\gamma_i^*\}$ when the condition $\Sigma_{j=1}^n f_j'\neq 1$ is satisfied.

Actually, consider an extreme case in which a particular set $\{\gamma_i^*\}_{i=1}^n$ is chosen to force $a_i=0$, for all $i=1,2,\ldots,n$, so that the characteristic equation (10) reduces to $\lambda^n=0$, which in turn makes all eigenvalues take zero values.

Notice that the conditions $a_i=0$, $i=1,2,\ldots,n$, implies that there exists at least one set of solutions $\{\gamma_i^*\}_{i=1}^n$ for the following linear-equation system:

$$f_i'\sum_{j\neq i}^{n}\gamma_j^*+(f_i'-1)\gamma_i^*=f_i',\quad i=1,2,\ldots,n,$$

or, in matrix form:

$$\mathbf{F}\cdot\gamma=\mathbf{f}, \tag{11}$$

where $\gamma=(\gamma_1^*,\gamma_2^*,\ldots,\gamma_n^*)^{\mathrm{T}}$, $\mathbf{f}=(f_1',f_2',\ldots,f_n')^{\mathrm{T}}$, and

$$\mathbf{F}=\begin{bmatrix}F_1\\F_2\\\vdots\\F_i\\\vdots\\F_n\end{bmatrix}=\begin{bmatrix} f_1'-1 & f_1' & \cdots & f_1' & \cdots & f_1'\\ f_2' & f_2'-1 & \cdots & f_2' & \cdots & f_2'\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ f_i' & f_i' & \cdots & f_i'-1 & \cdots & f_i'\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ f_n' & f_n' & \cdots & f_n' & \cdots & f_n'-1 \end{bmatrix}.$$

The sufficient condition for the existence of the solution $\boldsymbol{\gamma}$ of Eq. (11) is $|\mathbf{F}|\neq 0$. We shall further show that $|\mathbf{F}|\neq 0$ is equivalent to condition (9).

Apparently, summing up all row vectors of matrix $\mathbf{F}$ gives a null vector if $\Sigma_{j=1j}^n f_j'=1$.

On the other hand, if $|\mathbf{F}|=0$, there must exist a set of scalars $c_1, c_2, \cdots c_n$ (not all zeros) such that $c_1F_1+c_2F_2+\cdots c_nF_n=\mathbf{0}$, where $\mathbf{0}$ indicates a null vector, which implies that,

$$\sum_{j=1}^{n} c_j f_j' = c_i, \quad \text{for} \quad i=1,2,\ldots,n,$$

or, equivalently, $c_1=c_2=\cdots=c_n$, the latter in turn revealing that $\Sigma_{j=1}^{n} f_j'=1$.

Therefore, we can be ensured the existence of a unique $\gamma$ for Eq. (11) when condition (9) is met.

Furthermore, the continuity of function $f$ implies the existences of $\epsilon_j$ such that when $\gamma_i \in (\gamma_i^* - \epsilon_j, \gamma_i^* + \epsilon_j)$, $j=1,2,\ldots,n$, the maximum modulus of characteristic roots for Eq. (10) can be safely kept to be less than unity.

Next, we turn to prove the "only when" part, that is, the necessity of Eq. (9).

We proceed by showing that unity would be one of the characteristic roots of Eq. (10), should the condition (9) be violated.

In fact, if $\lambda=1$ is a solution of Eq. (10), substitution of $\lambda=1$ into Eq. (10) would lead to an equality: $\Sigma_{j=1}^{n} a_j=1$, which in turn demands that

$$\left(1-\sum_{j=1}^{n}\gamma_j\right)\sum_{j=1}^{n} f_j' + \sum_{j=1}^{n}\gamma_j = 1. \tag{12}$$

No matter what $\gamma_j$'s are taken, the equality of Eq. (12) is always achieved if $\Sigma_{j=1}^{n} f_j'=1$. That is, there is no way to improve the stability of the fixed point $\bar{y}$ when condition (9) is violated. Q.E.D.

To improve our understanding of the functioning of lagged adaptive adjustment, we shall then examine a simplest second-order discrete system in detail.

*Corollary 1 (A general second-order discrete system)*. For a general second-order discrete system $y_t=f(y_{t-1},y_{t-2})$, the lagged adaptive adjustment achieved through

$$\begin{aligned} y_t &= \tilde{f}(y_{t-1},y_{t-2}) \\ &= [1-(\gamma_1+\gamma_2)]f(y_{t-1},y_{t-2})+\gamma_1 y_{t-1}+\gamma_2 y, \end{aligned} \tag{13}$$

can force the system to converge to an unstable fixed point $\bar{y}$ if and only if the following conditions are met:

$$-[1-(\gamma_1+\gamma_2)]f_2'-\gamma_2<1, \tag{14}$$

$$[1-(\gamma_1+\gamma_2)](f_1'+f_2')+(\gamma_1+\gamma_2)<1, \tag{15}$$

$$[1-(\gamma_1+\gamma_2)](f_1'-f_2')+(\gamma_1-\gamma_2)>-1. \tag{16}$$

*Proof*. The Jacobian of Eq. (13) at any specified fixed point $\bar{y}$ is given by

$$\mathcal{J}(\bar{y})=\begin{pmatrix} 0 & 1 \\ [1-(\gamma_1+\gamma_2)]f_2'+\gamma_2 & [1-(\gamma_1+\gamma_2)]f_1'+\gamma_1 \end{pmatrix}.$$

Denote

$$\mathcal{T}=[1-(\gamma_1+\gamma_2)]f_1'+\gamma_1=\text{trace of } \mathcal{J},$$

$$\mathcal{D}=-[1-(\gamma_1+\gamma_2)]f_2'-\gamma_2=\text{determinant of } \mathcal{J},$$

then the associated eigenvalues for $\mathcal{J}(\bar{y})$ can be simply expressed as (see Ref. [2] for a detailed discussion):

$$\lambda_{1,2}=\frac{1}{2}(\mathcal{T}\pm\sqrt{\mathcal{T}^2-4\mathcal{D}}).$$

The local convergency of the system to a particular fixed point $\bar{y}$ (that is, the local stability of $\bar{y}$) is guaranteed if and only if the following three inequalities hold simultaneously:

$$\left.\begin{aligned} \mathcal{D}&<1 \\ \mathcal{T}-D&<1 \\ \mathcal{T}+D&>-1 \end{aligned}\right\}.$$

Substituting $\mathcal{T}=[1-(\gamma_1+\gamma_2)]f_1'+\gamma_1$ and $\mathcal{D}=-[1-(\gamma_1+\gamma_2)]f_2'-\gamma_2$ will then give us the conditions (14)–(16). Q.E.D.

Therefore, the three inequalities (14)–(16) form an effective region in the adaptive parameters $(\gamma_1,\gamma_2)$ space, inside which, all $\{\gamma_1,\gamma_2\}$ combinations ensure the stability of $\bar{y}$.

## IV. NUMERICAL SIMULATIONS

*Example 1 (Delayed logistic system)*. Consider the following second-order discrete system:

$$\begin{aligned} y_t &= \theta(y_{t-1},y_{t-2}) \\ &= 4ay_{t-1}(1-y_{t-1})+4(1-a)y_{t-2}(1-y_{t-2}), \end{aligned} \tag{17}$$

where $1>a>0$. This system is a higher-order version of the famous logistic equation and hence gives a unique nontrivial fixed point $\bar{y}=3/4$, which is stable when Eqs. (14)–(16) hold for $\gamma_1=\gamma_2=0$, which leads to $1/2<a<3/4$. Therefore, either when $a\in A_1=(0,1/2)$ or $a\in A_2=(3/4,1)$, the nontrivial fixed point $\bar{y}=3/4$ becomes unstable. It is worthwhile to notice that the asymptotic dynamical behavior in long term are totally different when $a\in A_1$ and $a\in A_2$. This can be seen from the shapes of chaotic strange attractors depicted in the $(y_t,y_{t-1})$-plane, as shown in Fig. 2. It is easy to verify that $\theta_1'(\bar{y})=-2a$ and $\theta_2'(\bar{y})=-2(1-a)$, which gives

$$\left.\begin{aligned} \theta_1'-\theta_2'&=2-4a \\ \theta_1'+\theta_2'&=-2 \end{aligned}\right\},$$

therefore, condition (9) is not violated and according to Theorem 1 the implementation of a lagged adaptive adjustment mechanism by

$$\begin{aligned} y_t=\tilde{\theta}(y_{t-1},y_{t-2}) &= (1-\gamma_1-\gamma_2)[4ay_{t-1}(1-y_{t-1}) \\ &\quad +4(1-a)y_{t-2}(1-y_{t-2})]+\gamma_1 y_{t-1}+\gamma_2 y_{t-2}, \end{aligned}$$

should be an effective and efficient way to stabilize the system.

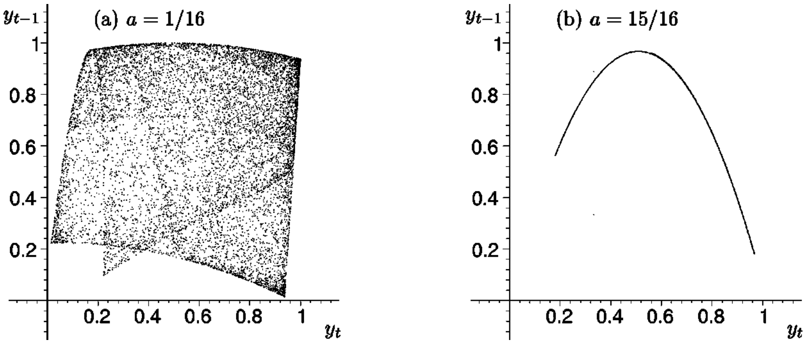


FIG. 2. Strange attractors of $\theta$ ($y_0 = y_1 = 0.3333$).

The conditions (14)–(16) become

$$\begin{array}{l} \gamma_2 > \dfrac{1}{3-2a}[1-2a-2(1-a)\gamma_1] \\ \gamma_2 < 1-\gamma_1 \\ (3-4a)\gamma_2 < 3-4a+(4a-1)\gamma_1 \end{array}. \qquad (18)$$

Now we examine two particular values of $a$, one in $A_1$ and the other in $A_2$.

*Case I*, $a = 1/16$. Condition (18) reduces to

$$\begin{array}{l} \gamma_2 > \dfrac{7}{23} - \dfrac{15}{23}\gamma_1 \\ \gamma_2 < 1-\gamma_1 \\ \gamma_2 < 1 - \dfrac{3}{11}\gamma_1 \end{array}.$$

*Case II*, $a = 15/16$. Condition (18) reduces to

$$\begin{array}{l} -\dfrac{7}{9} - \dfrac{1}{9}\gamma_1 < \gamma_2 \\ \gamma_2 < 1-\gamma_1 \\ \gamma_2 > 1 - \dfrac{11}{3}\gamma_1 \end{array}.$$

The effective regions for the feedback parameters $\gamma_1$ and $\gamma_2$ formed by the above inequalities are depicted in Fig. 3. While Fig. 4(a) illustrates a typical trajectory of a delayed logistic system for $a = 1/16$, two stabilized trajectories starting at the same initial points as in Fig. 2(a) and Fig. 4(a) are shown in Figs. 4(b) and 4(c) for different feedback parameters. One qualitative difference is observed between the effective regions shown in Figs. 3(a) and 3(b). While in the former case ($a \in A_1$), the effective region covers parts of the $\gamma_1$ and $\gamma_2$ axes, which means that a lagged adaptive adjustment from either $y_{t-1}$ or $y_{t-2}$ alone is sufficient to stabilize the original system. But for the latter case ($a \in A_2$), the effective region only covers parts of the $\gamma_1$ axis but not the $\gamma_2$ axis, which means that a lagged adaptive adjustment from either $y_{t-2}$ alone is insufficient to stabilize the original system [7]. This is shown in Figs. 5(b) and 5(c). While lagged adaptive adjustment from $y_{t-1}$ alone does stabilize the trajectory in a few iterations, lagged adaptive adjustment from $y_{t-2}$ alone fails to do so.

## V. UNIFORMLY LAGGED ADAPTIVE ADJUSTMENT

It is also observed that in Fig. 2, the diagonal lines ($\gamma_1 = \gamma_2$) pass the effective regions for both cases, which means that the stabilization can always be achieved by uniformly lagged adaptive adjustment.

*Definition 3*. By a *uniformly* lagged adaptive adjustment, we mean $\gamma_j = \gamma/n > 0$ for all $j = 1,2,\ldots,n$ in Eq. (7), that is,

$$\begin{aligned} y_t &= \tilde{f}_\gamma(y_{t-1}, y_{t-2}, \ldots, y_{t-n}) \\ &= (1-n\gamma) f(y_{t-1}, y_{t-2}, \ldots, y_{t-n}) + \gamma \sum_{j=1}^{n} y_{t-j}. \end{aligned} \qquad (19)$$

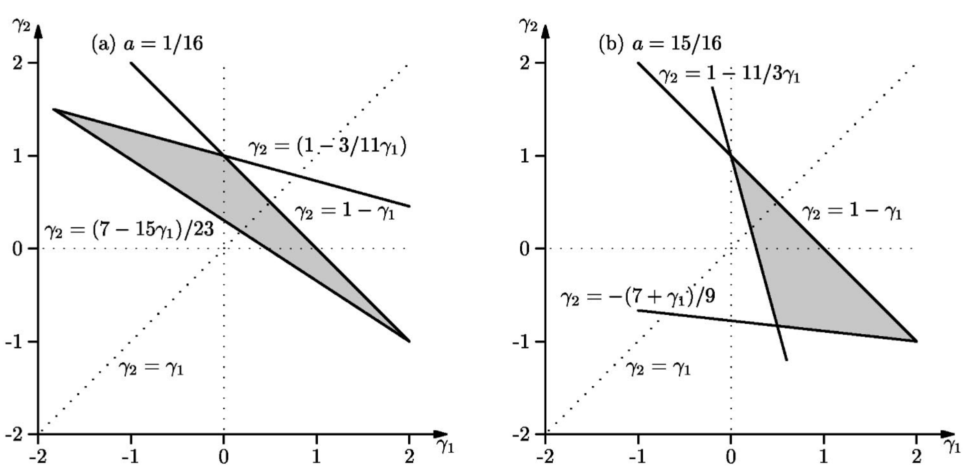


FIG. 3. Effective regions of feedback parameters.

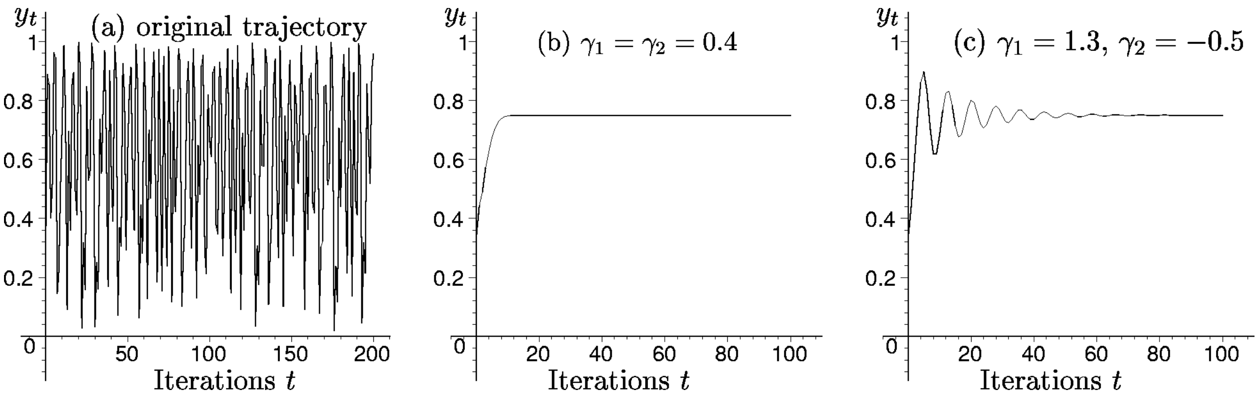


FIG. 4. Numerical simulations for $\theta$ and $\tilde{\theta}$, $a=1/16$ $(y_0=y_1=0.3333)$.

By noticing that $y_t \to \bar{y}$ if $\gamma \to 1/n$, we may expect that the results stated in Theorem 1 still apply, that is, condition (9) is both a necessary and sufficient condition for the functioning of lagged adaptive adjustment. It is a pity that we are still unable to verify such intuition in theory. Nevertheless, for a second-order discrete system, we do arrive at the following conclusion:

*Corollary 2.* For a second-order discrete dynamical system defined by $y_t = f(y_{t-1}, y_{t-2})$, *if* $\sum_{j=1}^{n} f_1' + f_2' \neq 1$ holds at an unstable fixed point, there always exists an interval $\Gamma = (1/2-\alpha, 1/2+\beta)$, $\alpha,\beta \geqslant 0, +\beta), \alpha,\beta \geqslant 0$, such that the unstable fixed point $\bar{y}$ can be stabilized through a uniformly lagged adaptive adjustment defined by Eq. (19) when $\gamma \in \Gamma$.

*Proof.* For a second-order discrete system, a uniformly lagged adaptive adjustment amounts to $\gamma_1 = \gamma_2 = \gamma > 0$, and, therefore, conditions (14)–(16) become

$$-(1-2\gamma) f_2' - \gamma < 1, \tag{20}$$

$$(1-2\gamma)(f_1' + f_2') + 2\gamma < 1, \tag{21}$$

$$(1-2\gamma)(f_1' - f_2') > -1. \tag{22}$$

As long as $f_1' + f_2' \neq 1$, the stabilization can be ensured through letting $2\gamma$ approach unity either from below ($\gamma = 1/2 - \epsilon$, $\epsilon \to 0$) or from above ($\gamma = 1/2 + \epsilon$, $\epsilon \to 0$). Q.E.D.

For the delayed logistic system of Example 1, since $\theta_1' + \theta_2' = -2 < 1$ holds for all $a$, the system (17) can always be stabilized through a uniform adjustment for any $a$.

It is rather difficult to derive a set of conditions like Eqs. (20)–(22) for a discrete system with order higher than two. But the following example does demonstrate the possibilities.

*Example 2 (A delayed Hennon map*). Consider the following third-order discrete system:

$$x_t = g(x_{t-1}, x_{t-2}, x_{t-3}) = \alpha\left(\frac{7}{5} - x_{t-1}^2 + \frac{3}{10}x_{t-2}\right) + (1-\alpha)\left(\frac{7}{5} - x_{t-2}^2 + \frac{3}{10}x_{t-3}\right).$$

This is a third-order version of the Hennon map and hence gives a pair of fixed points

$$\bar{x}_{1,2} = -\frac{7}{20} \pm \frac{1}{20}\sqrt{609} = \{0.8839, -1.5839\}.$$

Simple mathematical manipulations reveal that

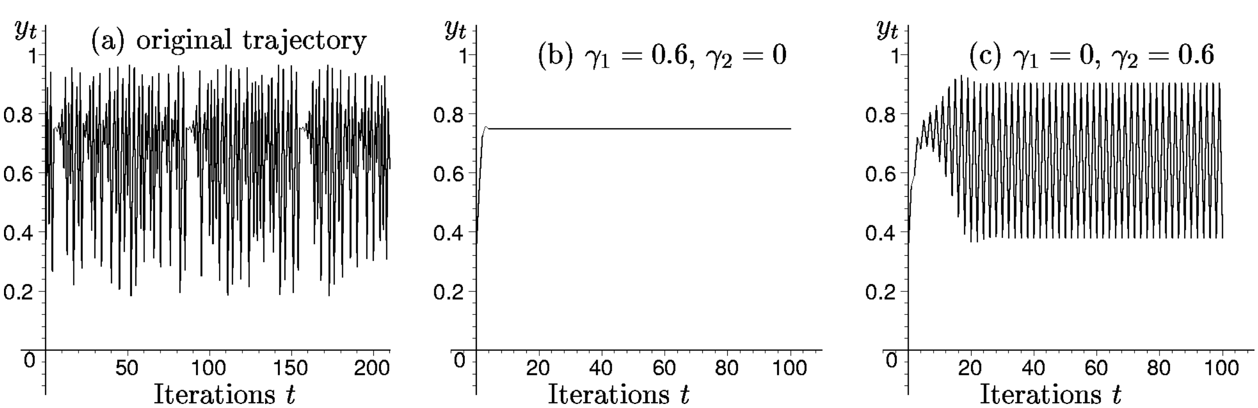


FIG. 5. Numerical simulations for $\theta$ and $\tilde{\theta}$, $a=15/16$ $(y_0=y_1=0.3333)$.

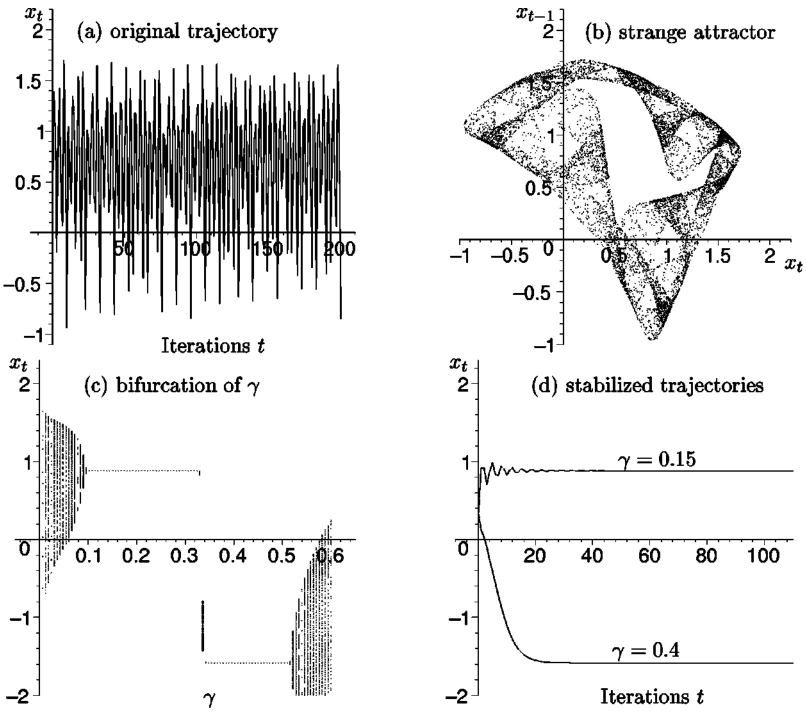


FIG. 6. Numerical simulations of $g$ and $\tilde{g}$, $a = 4/5$ $(x_0 = x_1 = x_2 = 0.3333)$.

$$\frac{\partial g}{\partial x_{t-1}} = -2\alpha x_{t-1}$$

$$\frac{\partial g}{\partial x_{t-2}} = \frac{3}{10}\alpha - 2(1-\alpha)x_{t-2},$$

$$\frac{\partial g}{\partial x_{t-2}} = \frac{3}{10}(1-\alpha)$$

so that at any fixed point $\bar{x}$,

$$g_1' + g_2' + g_3' = \frac{3}{10} - 2\bar{x},$$

$$g_1' - g_2' + g_3' = (1-2\alpha)\left(\frac{3}{10} + 2\bar{x}\right).$$

Therefore, for both $\bar{x}_1$ and $\bar{x}_2$, we have $g_1' + g_2' + g_3' \neq 1$ and $g_1' - g_2' + g_3' \neq -1$.

Figures 6(a) and 6(b) show a typical trajectory and the strange attractor formed for the case of $\alpha = 0.8$ A uniformly lagged adjustment is implemented as

$$x_t = \tilde{g}(x_{t-1}, x_{t-2}, x_{t-3})$$
$$= (1-3\gamma)g(x_{t-1}, x_{t-2}, x_{t-3}) + \gamma(x_{t-1} + x_{t-2} + x_{t-3}).$$

Figure 6(c) shows a bifurcation diagram for the adjustment parameter $\gamma$. We see that, when $0.1 < \gamma < 0.33$, the system is stabilized to $\bar{x}_1$, but when $0.33 < \gamma < 0.5$, the system is stabilized to $\bar{x}_2$. Two typical trajectories are overlapped in Fig. 6(d).

## VI. CONCLUDING REMARKS

It needs to be mentioned that the idea of controlling chaos or unstable periodic orbits through feedbacks is not new to physicists. For a chaotic discrete system $x_t = f(x_{t-1})$, Pyragas [3] discussed the possibility of applying a delayed feedback control in the form of

$$x_t = f(x_{t-1}) + k(x_{t-1} - x_{t-2}), \tag{23}$$

where $k$ is an arbitrary constant. Socolar *et al.* [6] extended the Pyragas method to include, in the control term, memory of all the previous states of the system (which may seem to be in common with our mechanism in intuition). Similarly experimental studies have been found in other literature listed in Refs. [5] and [6].

Pyragas' delayed feedback (23) is originally designed for the first-order discrete system. Although adaptive adjustment mechanism resembles Pyragas scheme in appearance, there does not exist any strict relationship between them. That is, neither of them can be expressed as a special case of the other. The implementation of Pyragas' method increases the dimensionality of the original system, which may not guarantee in theory that an original stable system can still remain "stable." The lagged adaptive adjustment mechanism proposed in this paper, however, does ensure that all originally stable systems remain stable, and by suitably adjusting feedback parameters from zero onwards, a chaotic system would definitely converge to its generic fixed points, should not the condition (9) be violated. It is easy to implement and achieves convergence at a rather high speed. Besides, it requires neither a *priori* information about the system itself nor any externally generated control signal.

Finally, the (lagged) adaptive adjustment mechanism can be similarly applied to stabilize spatiotemporal chaos in coupled map lattice systems. The issue of how to apply the technique to real systems where only global variables rather than local variables are observed (such as discussed in Ref. [6]) will be an important subject of further research.

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[7] This phenomenon is expected to be generically true for all $a \in A_2$. Detailed discussion for adaptive feedback involving only parts of delayed variables will be provided in a follow-up paper.

[8] The term ‘‘adaptive’’ usually implies a somehow self-adjusting parameter change during application of the method. The misleading name ‘‘adaptive adjustment’’ is adopted just to follow the convention of ‘‘adaptive expectation’’ that has been used in economics since 1950’s.